Schwartz 7.4 (a)

$$H_0 = \frac{1}{2}\phi\Box\phi \qquad H_{int} = \frac{1}{2}m^2\phi^2$$

$O(1)$: $\quad 1$ —— 2

$O(m^2)$: $\quad 1$ —— x —— 2

$O(m^4)$: $\quad 1$ —— x_1 —— x_2 —— 2

$O(m^6)$: $\quad 1$ —— x_1 —— x_2 —— x_3 —— 2.

Schwartz
7.4 (c) this also includes part (b)

\# 0 th order: $\langle 0 | T\{ \phi_1 \phi_2 \} | 0 \rangle$

$= D_{12}$

Apply LSZ: $(-i)^2 \int d^4x_1\, e^{-ip_1 x_1} p_1^2 \int d^4x_2\, e^{ip_2 x_2} p_2^2\, D_{12}$

Let the integral be implicit.

$$= -e^{-ip_1x_1} e^{ip_2x_2} \frac{i}{k^2 + i\varepsilon} e^{ik(x_1-x_2)} p_1^2 p_2^2$$

$$= (-i) \frac{e^{ix_1(k-p_1)} e^{ix_2(p_2-k)}}{k^2 + i\varepsilon} p_1^2 p_2^2$$

$$= (-i) \frac{e^{ix_2(p_2-p_1)}}{p_1^2 + i\varepsilon} p_1^2 p_2^2$$

$$= \boxed{(-i) p_2^2\, \delta(p_2 - p_1)}$$

* 1st order: $\frac{-im^2}{2!} \langle 0 | T\{\phi_1 \phi_2 \phi_\alpha^2\} | 0 \rangle$

$$= -im^2 \int d^4\alpha \; D_{1\alpha} D_{\alpha 2}$$

Again, let the integral over undetermined position and momenta be implicit.

$$= -im^2 \frac{i}{k_1^2 + i\varepsilon} e^{ik_1(x_1 - \alpha)} \frac{i}{k_2^2 + i\varepsilon} e^{ik_2(\alpha - x_2)}$$

$$= \frac{im^2 \, e^{i\alpha(k_2 - k_1)} e^{ik_1 x_1} e^{ik_2 x_2}}{(k_1^2 + i\varepsilon)(k_2^2 + i\varepsilon)}$$

$$= im^2 \frac{e^{ik_1(x_1 + x_2)}}{(k_1^2 + i\varepsilon)(k_1^2 + i\varepsilon)}$$

Apply LSZ, multiply by $(-i e^{-ip_1 x_1} p_1^2)(-i e^{ip_2 x_2} p_2^2)$ and integrate over x_1, x_2, again, the integration is implicit

$$= \frac{im^2(-1) \, e^{ik_1(x_1 + x_2)} e^{-ip_1 x_1} e^{ip_2 x_2} \; p_1^2 p_2^2}{(k_1^2 + i\varepsilon)^2}$$

$$= \frac{(-im^2) \, e^{ix_1(k_1 - p_1)} e^{ix_2(k_1 - p_2)}}{(k_1^2 + i\varepsilon)^2}$$

$$= \frac{(-im^2)\; e^{ix_2(p_1 - p_2)} \; p_1^2 p_2^2}{(p_1^2 + i\varepsilon)^2}$$

$$= \boxed{(-im^2) \frac{p_2^2}{p_1^2} \delta(p_1 - p_2)}$$

* 2nd order: $\frac{1}{2!}\left(\frac{-im^2}{2!}\right)^2 \langle 0 | T\{\phi_1 \phi_2 \phi_\alpha^2 \phi_\beta^2\} | 0 \rangle$

$$= (-im^2)^2 \int d^4\alpha \, d^4\beta \; D_{1\alpha} D_{\alpha\beta} D_{\beta 2}$$

$$\Downarrow$$

$$\frac{i}{k_1^2 + i\varepsilon} \frac{i}{k_2^2 + i\varepsilon} \frac{i}{k_3^2 + i\varepsilon} e^{ik_1(x_1 - \alpha)} e^{ik_2(\alpha - \beta)} e^{ik_3(\beta - x_2)}$$

$$= \frac{-i}{(\cdots)} e^{i\alpha(k_2 - k_1)} e^{i\beta(k_3 - k_2)} e^{ik_1 x_1} e^{-ik_3 x_2}$$

$$= \frac{-i}{(k^2 + i\varepsilon)^3} e^{ik(x_1 - x_2)}$$

Apply LSZ: multiply $\left(-i e^{-ip_1 x_1} p_1^2\right)\left(-i e^{ip_2 x_2} p_2^2\right)$

$$= \frac{(-i)^3 \; e^{ix_1(k - p_1)} e^{ix_2(p_2 - k)} \; p_1^2 p_2^2}{(k^2 + i\varepsilon)^3}$$

$$= i \frac{e^{i x_2 (p_2 - p_1)}}{(p_1^2 + i\varepsilon)^3} p_1^2 p_2^2$$

$$= i \frac{p_2^2}{(p_1^2)^2} \delta(p_2 - p_1)$$

Multiply back $(-im^2)^2$, we have

$$\boxed{(-im^2)^2 i \frac{p_2^2}{(p_1^2)^2} \delta(p_2 - p_1)}$$

* 3rd order: $\frac{1}{3!}\left(\frac{-im^2}{2!}\right)^3 \langle 0 | T\{\phi_1 \phi_2 \phi_\alpha^2 \phi_\beta^2 \phi_\gamma^2\} | 0 \rangle$

multiplicity is $6 \times 4 \times 2 = 48$, cancels with $\frac{1}{3!}\left(\frac{1}{2!}\right)^3$.

$$= (-im^2)^3 \int d^4\alpha \, d^4\beta \, d^4\gamma \, D_{1\alpha} D_{\alpha\beta} D_{\beta\gamma} D_{\gamma 2}$$

$$\Downarrow$$

$$= \frac{i}{k_1^2 + i\varepsilon} \frac{i}{k_2^2 + i\varepsilon} \frac{i}{k_3^2 + i\varepsilon} \frac{i}{k_4^2 + i\varepsilon} e^{ik_1(x_1 - \alpha)} e^{ik_2(\alpha - \beta)} e^{ik_3(\beta - \gamma)} e^{ik_4(\gamma - x_2)}$$

$$= (\ldots) e^{i\alpha(k_2 - k_1)} e^{i\beta(k_3 - k_2)} e^{i\gamma(k_4 - k_3)} e^{ik_1 x_1} e^{-ik_4 x_2}.$$

$$= \frac{1}{(k^2 + i\varepsilon)^4} e^{ik(x_1 - x_2)}$$

Apply LSZ: multiply by $\left(-i e^{-ip_1 x_1} p_1^2\right)\left(-i e^{ip_2 x_2} p_2^2\right)$

$$= \frac{(-1)}{(k^2+i\varepsilon)^4} e^{ix_1(k-p_1)} e^{ix_2(p_2-k)} p_1^2 p_2^2$$

$$= \frac{-1}{(p_1^2+i\varepsilon)^4} e^{ix_2(p_2-p_1)} p_1^2 p_2^2$$

$$= \frac{-p_2^2}{(p_1^2)^3} \delta(p_2-p_1)$$

putting back $(-im^2)^3$, we have

$$\boxed{= (-im^2)^3 \frac{p_2^2}{(p_1^2)^3} \delta(p_2-p_1)}$$

$O(1)$: $(-i)\, p_2^2 \;\; \delta(p_2 - p_1)$

$O(m^2)$: $(-i m^2) \dfrac{p_2^2}{p_1^2} \; \delta(p_2 - p_1)$

$O(m^4)$: $(-i m^2)^2 \, i \dfrac{p_2^2}{(p_1^2)^2} \; \delta(p_2 - p_2)$

$O(m^6)$: $-(-i m^2)^3 \dfrac{p_2^2}{(p_1^2)^3} \; \delta(p_2 - p_1)$

Adding:

$$\mathcal{M} \propto p_2^2 \, \delta(p_2 - p_1) \left[-i + \frac{-i m^2}{p_1^2} + \frac{i(-i m^2)^2}{(p_1^2)^2} + \frac{-(-i m^2)^3}{(p_1^2)^3} + \cdots \right]$$

$$= \delta(p_2 - p_1)(-i p_2^2) \left[1 + \frac{m^2}{p_1^2} + \frac{m^4}{p_1^4} + \frac{m^6}{p_1^6} \right]$$

$$= \delta(p_2 - p_1)(-i p_2^2) \left[\frac{p_1^2}{p_1^2 - m^2} \right]$$

$$= \int d^4x \, \frac{e^{ix(p_2 - p_1)} (-i)\, p_1^2 p_2^2}{p_1^2 - m^2}$$

$$= \int d^4x_1\, d^4x_2 \; \frac{e^{-ix_1(k-p_1)}\, e^{ix_2(p_2-k)}}{k^2-m^2} (-i) p_1^2\, p_2^2$$

$$= (-i)^2 \int d^4x_1 \int d^4x_2 \; e^{-ix_1p_1}\, p_1^2\, e^{ix_2p_2}\, p_2^2 \; \frac{i}{k^2-m^2}\, e^{-ik(x_1-x_2)}$$

$$= \left[-i \int d^4x_1\, e^{-ix_1p_1}\, p_1^2\right] \left[-i \int d^4x_2\, e^{ix_2p_2}\, p_2^2\right] \frac{i}{k^2-m^2}\, e^{ik(x_1-x_2)}$$

One can clearly see the inversion of the LSZ reduction formula applied on $D_F(x_1, x_2)$.

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Schwartz
7.4 (d)

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)(\partial_\mu\phi)$$

$\Rightarrow$ EQM: $\Box\phi = 0$.

Solve it with a current J, which can be anything, such as delta function,

$$\Box\phi = J, \quad \phi = \frac{1}{\Box}J.$$

Perturb it: $\mathcal{L}_{int} = -\frac{1}{2}m^2\phi^2$, then

$$\mathcal{L}_0 + \mathcal{L}_{int} = -\frac{1}{2}(\partial_\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m^2\phi^2,$$

EQM: $(\Box + m^2)\phi = 0$, $\Rightarrow$ $(\Box + m^2)\phi = J$.

Assuming $\phi = \phi_0 + \phi_1$, where $\phi_0 = \frac{1}{\Box}J$, we now wish to solve for ϕ_1 in $O(m^2)$

$$(\Box + m^2)\phi = J, \quad (\Box + m^2)(\phi_0 + \phi_1) = J,$$

$$\cancel{\Box\phi_0} + m^2\phi_0 + \Box\phi_1 + m^2\phi_1 = \cancel{J},$$

$$m^2\phi_0 + \Box\phi_1 + O(m^4) = 0,$$

$$\phi_1 = -\frac{m^2}{\Box}\phi_0,$$

$$\Rightarrow \quad \phi = \frac{1}{\Box}J - \frac{m^2}{\Box^2}J + O(m^4)$$

To next order, Assume $\phi_0 = \frac{1}{\Box} J - \frac{m^2}{\Box^2} J$,

we want to solve for ϕ' in $\phi = \phi_0 + \phi'$ to $O(m^4)$

$$(\Box + m^2)\phi = (\Box + m^2)\left(\frac{1}{\Box} J - \frac{m^2}{\Box^2} J + \phi'\right) = J$$

$$J - \frac{m^2}{\Box} J + \Box\phi' + \frac{m^2}{\Box} J - \frac{m^4}{\Box^2} J + m^2\phi' = J$$

$$\Box\phi' - \frac{m^4}{\Box^2} J + O(m^6) = 0,$$

$$\phi' = \frac{m^4}{\Box^3} J.$$

We have, to $O(m^4)$:

$$\phi = \frac{1}{\Box} J - \frac{m^2}{\Box^2} J + \frac{m^4}{\Box^3} J + \cdots$$

$$= \frac{1}{\Box}\left(1 - \frac{m^2}{\Box} + \left(\frac{m^2}{\Box}\right)^2 + \cdots\right) J.$$

$$= \frac{1}{\Box}\left(\frac{1}{1 - \frac{-m^2}{\Box}}\right) J$$

$$= \frac{1}{\Box}\left(\frac{\Box}{\Box + m^2}\right) J$$

$$= \frac{1}{\Box + m^2} J,$$

Replacing $\Box$ with $-k^2$, we obtain

$$\phi = \frac{1}{-k^2 + m^2} J = \boxed{\frac{-1}{k^2 - m^2} J.}$$

This solves $(\Box + m^2)\phi = J$, the full interaction EOM.

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3.15.2024